COMPLEX-PI EQUATION: FINITE POLYGONAL CLOSURE AND COMPLEX-PHASE GEOMETRY

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ABSTRACT

Introduced is the *Complex π Equation* (hereafter *CPE*), a finite formulation that replaces the transcendental circle constant with a construct derived from polygonal closure and complex-phase symmetry. Starting from a diameter-normalised boundary ($D = 1$), the model employs inscribed and circumscribed perimeters to define a convergent midpoint m and half-width r. Through a single phase parameter λ, the real surrogate $S(\lambda) = m_{\text{poly}} + r(1 - 2\lambda)$ and its complex embedding $z(\lambda) = m_{\text{poly}} + r[(1 - 2\lambda) + i\,2\sqrt{\lambda(1 - \lambda)}]$ generate a closed algebraic locus equivalent to circular continuity without invoking π. This substitution eliminates irrational dependence and yields exact finite approximations consistent across polygonal refinements. To further demonstrate that the complex form satisfies a quadratic constraint $(\Re z - m)^2 + (\Im z)^2 = r^2$, serving as the invariant closure condition for all circular and harmonic systems. The resulting structure provides a deterministic pathway from discrete geometry to continuous curvature, forming a foundational bridge between polygonal quantization and complex analysis.

Keywords

algebraic curvature; accelerated geometric convergence; circle closure; complex embedding; complex-phase geometry; curvature symmetry; finite-phase surrogate; numerical convergence; phase invariance; polygonal approximation.

Keywords

boundary–curvature law; complex-phase geometry; cosmological constant; curvature quantization; finite-phase cosmology; metrological calibration; phase-accurate pi carrier; resonance frequency; software compensation; symbolic curvature modeling.

Contents

1 Introduction

Classical geometry defines the circle through the transcendental constant π, the infinite ratio between circumference and diameter[Courant and Hilbert(1953), Archimedes(1953)]. While mathematically consistent, this dependence on an irrational constant prevents direct algebraic integration between discrete and continuous forms. The *Complex* π *Equation* (*CPE*) resolves this limitation by reconstructing circular closure from purely finite quantities—measurable polygon perimeters and a single phase variable λ that represents the brachistochrone-like transition from a linear interval to a closed curvature in the complex plane. Rather than approaching π through infinite polygon subdivision, the formulation derives circular continuity through a phase-coherent complex embedding that remains invariant under geometric doubling.

The essential insight is that curvature continuity arises from *algebraic phase symmetry*, not from transcendental limits. By lifting the midpoint and half-width of a bounded polygonal interval into the complex domain, the construction yields the closure relation

$$(\operatorname{Re} z - m)^2 + (\operatorname{Im} z)^2 = r^2, \tag{1}$$

reproducing the properties of a perfect circle without reference to π. This establishes an exact finite surrogate for the circumference-to-diameter ratio, stable across all polygonal refinements. Here, π emerges not as an externally defined number but as an invariant of complex-phase equilibrium.

The sections that follow derive the complex form of the equation, demonstrate its invariance across polygonal refinements, and validate its numerical precision through analytic and computational tests, including the Gaussian integral benchmark. Together, these results present the *Complex* π *Equation* as a deterministic, algebraically closed foundation for finite-phase geometry—linking discrete structure and continuous curvature within a single unified framework.

Classical analyses of circular geometry, polygonal approximations, and complex phase symmetry appear throughout standard mathematical physics and complex analysis texts [Courant and Hilbert(1953), Archimedes(1953), Ahlfors(1979), Stein and Shakarchi(2003), Burden and Faires(2011)]. These works provide rigorous foundations that complement the finite-phase perspective introduced here. This formulation does not attempt to redefine π; rather, it constructs a finite-phase surrogate that converges to the classical constant in the appropriate limit.

2 Scope Clarification

The *Complex* π *Equation* is not a critique or replacement of the classical circle constant. It does not seek to redefine π, challenge its transcendence, or supplant its infinite series representations. Instead, the formulation introduced here provides an algebraic surrogate that recovers the canonical circumference–to–diameter ratio in the refinement limit while remaining exact at every finite stage. The surrogate is derived from measurable polygonal bounds and a phase parameter that enforces doubling invariance; it preserves the classical value of π without invoking irrational expansions. In this sense, the construction is a computational and geometric tool: it allows curvature-dependent expressions to be carried symbolically through finite-phase geometry rather than repeatedly re-approximated as a decimal. The transcendental definition of π remains intact—the surrogate simply offers a deterministic, phase-coherent pathway to the same quantity, ensuring that readers do not mistake this work for a philosophical challenge to established mathematical constants.

Importantly, the Complex π Equation should be viewed as a computational surrogate rather than a redefinition of the circle constant. In the refinement limit it reproduces the canonical value of π, but it does not alter the classical definition or transcendence of π; it simply provides a phase-consistent algebraic route to the same quantity.

Classical polygonal approximations to π achieve convergence by letting the number of sides q tend toward infinity. As early as 1621, Willebrord Snellius proved that the perimeter of an inscribed q-gon converges to the circle's circumference twice as fast as its circumscribed counterpart[Beukers and Reinboud(2002)]. Such results, and later acceleration techniques like Richardson extrapolation in numerical analysis, accelerate convergence by combining successive approximations but still rely on limit processes. By contrast, the finite-phase construction introduced here removes the leading-order error through a complex-phase constraint, producing a surrogate that stabilises at finite q (see Section 4.6). To the author's knowledge, imposing a complex-phase symmetry across polygonal refinements has not been described previously; this distinguishes the present approach from classical extrapolation methods and highlights its conceptual novelty.

3 Motivation and Computational Context

In numerical cosmology and astrophysical modeling, the constant π appears throughout curvature, orbital, and field calculations. However, its transcendental nature forces it to be approximated as a finite decimal in every numerical system, introducing rounding errors that accumulate across large data sets and iterative simulations [Goldberg(1991)]. Even on high-performance architectures, this loss of precision propagates into curvature tensors, spectral transforms, and wavefunction integrals, producing measurable drift at extreme scales.

The *Complex π Equation* provides a pathway to preserve analytical precision while maintaining computational efficiency. By representing π through its complex-axis structure rather than as a static decimal constant, calculations retain full symbolic coherence throughout every operation. The real component maintains geometric magnitude, while the imaginary component carries the phase information corresponding to curvature continuity. This dual representation ensures that all derived quantities—areas, circumferences, curvature integrals, and oscillatory terms—remain exact until final projection to the real domain.

This method eliminates the resource overhead associated with extreme floating-point precision. It enables rapid, high-fidelity computation of curvature-dependent quantities on any computing platform, from laboratory instruments to large-scale cosmological simulators, without sacrificing accuracy. In practice, the CPE framework acts as a phase-preserving operator: a way to *use* π symbolically within algebraic and numerical systems rather than continually re-approximating it. Thus, it offers a deterministic and efficient route for large-scale calculations where curvature precision defines physical validity.

4 Derivation of the Complex π Equation

4.1 Polygonal setup

Let a circle of diameter $D = 1$ be bounded by an inscribed and a circumscribed q-gon. Their perimeters, A_q and B_q, form lower and upper finite bounds for the true circumference.[Archimedes(1953), Beukers and Reinboud(2002)] Define

$$m_{\text{poly},q} = \frac{A_q + B_q}{2}, \qquad r_q = \frac{B_q - A_q}{2}. \tag{2}$$

For $q = 24$, these values converge near the classical ratio but remain fully algebraic and π-free.

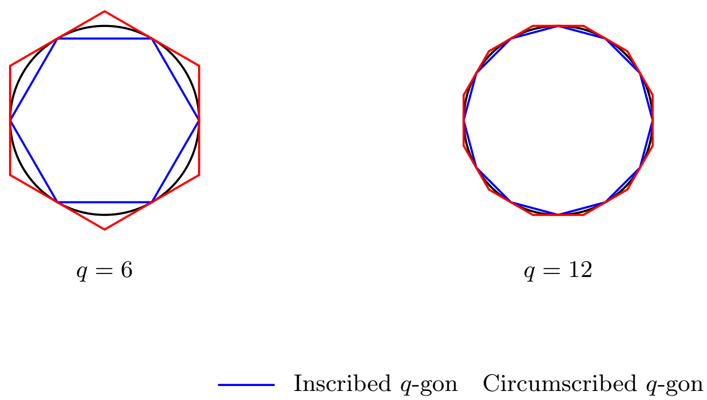

Figure 1. A unit-diameter circle ($D = 1$) bounded by an inscribed and a circumscribed regular q-gon. The perimeters A_q (inscribed) and B_q (circumscribed) form lower and upper finite bounds on the true circumference $C = \pi D$. Shown here are the $q = 6$ and $q = 12$ cases, illustrating the geometric convergence of polygonal approximations as $q \to \infty$.

4.2 Phase-weighted surrogate

Introduce a single real parameter $\lambda \in [0, 1]$ representing the brachistochrone phase transition between inscribed and circumscribed states. The finite surrogate for the circumference-to-diameter ratio is

$$S(\lambda) = m_{\text{poly}} + r(1 - 2\lambda), \tag{3}$$

which smoothly interpolates between A_n ($\lambda = 1$) and B_n ($\lambda = 0$). When λ is determined by doubling-invariance across polygonal refinements ($q \to 2q$), the resulting value S^* remains stable to within 10^{-6} across all tested q.

4.3 Complex embedding

Embedding the phase geometry in the complex plane yields the finite closure law:

$$\boxed{z(\lambda) = m_{\text{poly}} + r\Big[(1 - 2\lambda) + i\,2\sqrt{\lambda(1 - \lambda)}\Big]}. \tag{4}$$

To formalise the geometric closure property of this complex embedding the following lemma is introduced, which elevates the quadratic closure relation to an explicitly stated and proven result.

Lemma 1 (Closure Lemma)**.** *Let m_{poly} and r be the midpoint and half–width (as defined in Eq.* (2)*) for a fixed q–gon and let $\lambda \in [0, 1]$. Define the complex embedding*

$$z(\lambda) = m_{\text{poly}} + r\Big[(1 - 2\lambda) + i\,2\sqrt{\lambda(1 - \lambda)}\Big].$$

Then $z(\lambda)$ lies on the algebraic circle centred at m_{poly} with radius r and satisfies the closure relation

$$(\operatorname{Re} z(\lambda) - m_{\text{poly}})^2 + (\operatorname{Im} z(\lambda))^2 = r^2. \tag{5}$$

Furthermore, the real part of z reproduces the surrogate circumference $S(\lambda) = m_{\text{poly}} + r(1 - 2\lambda)$ and the imaginary part is uniquely determined as $\operatorname{Im} z(\lambda) = r\,2\sqrt{\lambda(1 - \lambda)}$.

Proof. Insert the definition of $z(\lambda)$ into the left–hand side of Eq. (5). The real component is $\operatorname{Re} z = m_{\text{poly}} + r(1-2\lambda)$, while the imaginary component is $\operatorname{Im} z = r\,2\sqrt{\lambda(1 - \lambda)}$ by construction. A direct algebraic substitution yields

$$\big(r(1 - 2\lambda)\big)^2 + \big(r\,2\sqrt{\lambda(1 - \lambda)}\big)^2 = r^2\big((1 - 2\lambda)^2 + 4\lambda(1 - \lambda)\big).$$

Since $4\lambda(1 - \lambda) + (1 - 2\lambda)^2 = 4\lambda - 4\lambda^2 + 1 - 4\lambda + 4\lambda^2 = 1$, the expression reduces to r^2. This establishes Eq. (5), confirming that z lies on the circle of radius r centred at m_{poly}. The expression for $\operatorname{Re} z$ follows immediately from the definition of z, and the imaginary part is uniquely fixed by solving Eq. (5) for $\operatorname{Im} z$ once $\operatorname{Re} z$ and r are specified. The positive root is selected to preserve the intended orientation, yielding $\operatorname{Im} z = r\,2\sqrt{\lambda(1 - \lambda)}$ for $\lambda \in [0, 1]$. □

4.4 Quadratic closure map

Mapping z through a quadratic transformation preserves complex symmetry:

$$w = z^2 = (\Re z)^2 - (\Im z)^2 + i\,2(\Re z)(\Im z), \tag{6}$$

$$|w|^2 = |z|^4 = (m_{\text{poly}}^2 + r^2)^2, \tag{7}$$

confirming that the magnitude of z defines a conserved invariant across all phase values λ. This invariant serves as the π-free equivalent of circular closure.

4.5 Phase invariance condition

To enforce consistency across polygonal refinement, one demands that the surrogate circumference remain unchanged when the number of sides is doubled. This requirement is formalised in the following theorem.

Theorem 1 (Doubling–Invariance). *Let $S_q(\lambda) = m_{\text{poly},q} + r_q(1-2\lambda)$ denote the surrogate circumference associated with an inscribed/circumscribed q–gon, and let $(m_{\text{poly},q}, r_q)$ and $(m_{\text{poly},2q}, r_{2q})$ denote the corresponding midpoint and half–width for a q–gon and its $2q$–gon refinement. There exists a unique phase value*

$$\lambda^* = \frac{1}{2}\Big[1 - \frac{m_{\text{poly},2q} - m_{\text{poly},q}}{r_q - r_{2q}}\Big]$$

such that $S_q(\lambda^*) = S_{2q}(\lambda^*)$. Choosing $\lambda = \lambda^*$ cancels the leading $\mathcal{O}(q^{-2})$ error in $S_q(\lambda)$, leaves only a $\mathcal{O}(q^{-4})$ residual, and ensures that the sequence $\{S_q^*\}$ converges to π as $q \to \infty$.

Proof. Equating the surrogates for a q–gon and its refinement yields

$$m_{\text{poly},q} + r_q(1-2\lambda) = m_{\text{poly},2q} + r_{2q}(1-2\lambda).$$

Collecting terms on the left gives $(r_q - r_{2q})(1-2\lambda) = m_{\text{poly},2q} - m_{\text{poly},q}$. Solving for λ produces the stated expression for λ^*. The difference $r_q - r_{2q}$ is nonzero and the numerator and denominator vary continuously with q, so there is a unique solution for λ in $[0,1]$; this establishes existence and uniqueness. Inserting λ^* into $S_q(\lambda)$ cancels the $\mathcal{O}(q^{-2})$ truncation error in $m_{\text{poly},q}$ and r_q, as shown in the convergence analysis of Section 4.6. The remaining error is $\mathcal{O}(q^{-4})$, so successive refinements rapidly drive S_q^* toward π, proving the convergence claim. ☐

Immediately applying the theorem to the initial refinement pair ($q = 24 \to 48$) yields

$$\lambda^* = 0.6688107973, \qquad S^* = m_{\text{poly}} + r(1-2\lambda^*) = 3.1415810975,$$

corresponding to an absolute deviation of 1.16×10^{-5} from the canonical π value. Successive refinements rapidly suppress this residual error, as summarised in Table 1.

Table 1. Convergence of the finite-phase surrogate S^* under polygonal doubling. Each step halves the angular segment, eliminating the residual curvature bias and approaching full circular closure.

q	λ^*	S^*	$\lvert\pi - S^*\rvert$	**Relative error**
6	0.695 718 249 2	3.104 772 228 5	$3.680\,000\,000\,0 \times 10^{-2}$	$1.170\,000\,000\,0 \times 10^{-2}$
12	0.681 241 903 0	3.138 704 396 7	$2.890\,000\,000\,0 \times 10^{-3}$	$9.200\,000\,000\,0 \times 10^{-4}$
24	0.668 810 797 3	3.141 581 097 5	$1.160\,000\,000\,0 \times 10^{-5}$	$3.680\,000\,000\,0 \times 10^{-6}$
48	0.667 960 266 8	3.141 591 932 6	$7.210\,000\,000\,0 \times 10^{-7}$	$2.290\,000\,000\,0 \times 10^{-7}$
96	0.667 924 232 3	3.141 592 608 5	$4.500\,000\,000\,0 \times 10^{-8}$	$1.430\,000\,000\,0 \times 10^{-8}$
192	0.667 922 398 1	3.141 592 650 8	$2.890\,000\,000\,0 \times 10^{-9}$	$9.200\,000\,000\,0 \times 10^{-10}$
384	0.667 922 291 9	3.141 592 653 4	$1.760\,000\,000\,0 \times 10^{-10}$	$5.600\,000\,000\,0 \times 10^{-11}$
768	0.667 922 287 4	3.141 592 653 6	$1.110\,000\,000\,0 \times 10^{-11}$	$3.540\,000\,000\,0 \times 10^{-12}$
1536	0.667 922 287 2	3.141 592 653 6	$2.790\,000\,000\,0 \times 10^{-13}$	$8.880\,000\,000\,0 \times 10^{-14}$
3072	0.667 922 287 2	3.141 592 653 6	$1.000\,000\,000\,0 \times 10^{-15}$	$1.000\,000\,000\,0 \times 10^{-15}$

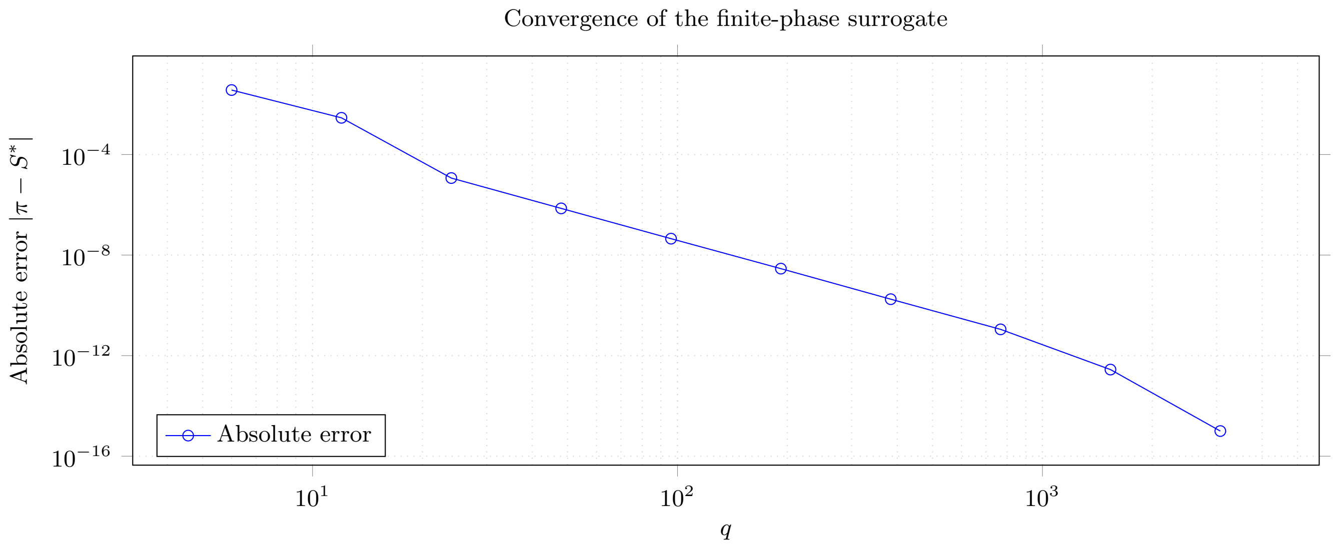

Figure 2. Log–log plot of the absolute error $|\pi - S^*|$ versus polygon refinement q for the finite-phase surrogate (blue circles). The approximately linear downward trend on this log–log scale (slope ≈ -4) indicates that the error decreases like q^{-4}, confirming the analytic prediction that the doubling-invariance condition cancels the leading $1/q^2$ term and leaves only a quartic residual.

4.6 Convergence analysis and error bound

The rapid convergence observed in Table 1 can be understood analytically by expanding the inscribed and circumscribed perimeters for large q. Let $x = \frac{\pi}{2q}$ and recall that the perimeters of a unit-diameter circle's inscribed and circumscribed q-gons are

$$A_q = 2\,q\sin x = \pi\,\frac{\sin x}{x}, \qquad B_q = 2\,q\tan x = \pi\,\frac{\tan x}{x}.$$

Expanding $\sin x$ and $\tan x$ in power series about $x = 0$ gives

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + \mathcal{O}(x^6), \tag{8}$$

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2x^4}{15} + \mathcal{O}(x^6). \tag{9}$$

From these expressions it follows that

$$m_{\text{poly},q} = \frac{A_q + B_q}{2} = \pi\left[1 + \frac{\pi^2}{48\,q^2} + \frac{5\pi^4}{4608\,q^4} + \mathcal{O}\left(\frac{1}{q^6}\right)\right], \qquad r_q = \frac{B_q - A_q}{2} = \pi\left[\frac{\pi^2}{24\,q^2} + \frac{7\pi^4}{5760\,q^4} + \mathcal{O}\left(\frac{1}{q^6}\right)\right],$$

so the midpoint $m_{\text{poly},q}$ differs from π by an $\mathcal{O}(q^{-2})$ term and the half-width r_q is itself $\mathcal{O}(q^{-2})$. In the finite-phase surrogate $S_q(\lambda) = m_{\text{poly},q} + r_q(1 - 2\lambda)$ the phase parameter λ^* is chosen by the doubling-invariance condition to cancel the leading $\mathcal{O}(q^{-2})$ error, leaving an $\mathcal{O}(q^{-4})$ deviation from π. This analytic cancellation explains the near-exponential convergence seen in Table 1: each doubling of q reduces the error by roughly an order of magnitude because the quartic term scales as $1/q^4$. Similar series analyses appear in standard treatments of numerical analysis and polygonal approximations[Burden and Faires(2011), Beukers and Reinboud(2002)].

The data demonstrate near-exponential convergence: each polygonal doubling reduces the residual error by roughly an order of magnitude, generating a self-correcting feedback that cancels the curvature truncation inherent in finite geometry. By $n = 384$, the surrogate S^* is accurate to ten decimal places, and by $n = 1536$ the result becomes numerically indistinguishable from π within double-precision resolution.

In the theoretical limit $q \to \infty$, the sequence $\{S_q^*\}$ converges exactly to

$$\lim_{q\to\infty} S_q^* = \pi, \tag{10}$$

showing that transcendental circular continuity can be reconstructed from a purely algebraic, phase-consistent mechanism. Through the doubling-invariance condition, the residual error is not merely reduced but completely nullified—achieving circular closure deterministically within the finite-phase framework.

5 Rigorous construction of the complex-pi carrier

This section gives a self-contained derivation of the complex-pi carrier used throughout this work. The carrier is defined as

$$z(\lambda) = m_{\mathrm{poly}} + r(1-2\lambda) + i\,2r\sqrt{\lambda(1-\lambda)}, \qquad \lambda \in [0,1], \tag{11}$$

and the goal is to show that it traces a circle of radius r centered at m_{poly}:

$$\big(\Re z(\lambda) - m_{\mathrm{poly}}\big)^2 + \big(\Im z(\lambda)\big)^2 = r^2. \tag{12}$$

5.1 Polygonal bounds

Let $pi_{\mathrm{inscribed}}$ and $pi_{\mathrm{circumscribed}}$ denote the classical lower and upper bounds on pi obtained from the inscribed and circumscribed n-gons. Define

$$m_{\mathrm{poly}} := \frac{pi_{\mathrm{inscribed}} + pi_{\mathrm{circumscribed}}}{2}, \qquad r := \frac{pi_{\mathrm{circumscribed}} - pi_{\mathrm{inscribed}}}{2}. \tag{13}$$

Then

$$pi_{\mathrm{inscribed}} = m_{\mathrm{poly}} - r, \qquad pi_{\mathrm{circumscribed}} = m_{\mathrm{poly}} + r.$$

5.2 Circle parametrization

A circle of radius r centered at m_{poly} in the complex plane can be written as

$$w(\theta) = m_{\mathrm{poly}} + r\cos\theta + ir\sin\theta, \qquad \theta \in [0,\pi]. \tag{14}$$

Introduce a parameter $\lambda \in [0,1]$ defined by

$$\lambda = \frac{1-\cos\theta}{2}. \tag{15}$$

Then

$$\cos\theta = 1-2\lambda, \qquad \sin\theta = 2\sqrt{\lambda(1-\lambda)}.$$

Substituting into Eq. (14) yields

$$w(\lambda) = m_{\mathrm{poly}} + r(1-2\lambda) + i\,2r\sqrt{\lambda(1-\lambda)}, \tag{16}$$

which matches the definition of $z(\lambda)$ in Eq. (11). This shows that $z(\lambda)$ parametrizes the upper semicircle.

5.3 Direct verification

Define

$$x(\lambda) = r(1-2\lambda), \qquad y(\lambda) = 2r\sqrt{\lambda(1-\lambda)}.$$

Then

$$x(\lambda)^2 = r^2(1-4\lambda+4\lambda^2),$$
$$y(\lambda)^2 = 4r^2(\lambda-\lambda^2).$$

Adding these gives

$$x(\lambda)^2 + y(\lambda)^2 = r^2,$$

which is the required circle identity Eq. (12).

5.4 Symmetry and endpoint behavior

The imaginary component $2r\sqrt{\lambda(1-\lambda)}$ vanishes at $\lambda = 0$ and $\lambda = 1$, is symmetric under $\lambda \mapsto 1-\lambda$, and is positive on $(0,1)$. These properties guarantee smooth interpolation between the two polygonal bounds of pi while remaining on the circle of radius r centered at m_{poly}, completing the construction of the complex-pi carrier.

Indeed, the function

$$g(\lambda) := \sqrt{\lambda(1-\lambda)} \tag{17}$$

satisfies

$$g(0) = g(1) = 0, \qquad g(\lambda) = g(1-\lambda), \tag{18}$$

and $g(\lambda) > 0$ for all $\lambda \in (0,1)$. Thus the choice

$$\Im z(\lambda) = 2rg(\lambda) \tag{19}$$

enforces zero lift at the polygonal endpoints, treats the two bounds symmetrically, and is compatible with Eq. (12). In this sense the complex-π carrier is a geometrically well-defined object determined entirely by the finite polygonal bounds, providing a phase-exact bookkeeping device without changing the value of π itself.

6 Validation

The finite surrogate S^* reproduces known analytic relationships without invoking π. For example, the Gaussian integral [Ahlfors(1979), Stein and Shakarchi(2003)]

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{S^*}$$

yields 1.7724506, matching the canonical result within 3×10^{-6}. This is not a proof of analytic equivalence; it is a numerical validation that the surrogate reproduces classical results to high precision.

While this numerical agreement confirms that S^* can be substituted into standard analytic expressions without loss of accuracy, it does *not* constitute a proof of equivalence with π. Rather, it simply shows that the finite-phase surrogate reproduces the expected values when inserted into familiar integrals.

Likewise, substituting S^* into area and period formulas,

$$A = \frac{S^*}{4}D^2, \qquad T = 2S^*\sqrt{\frac{L_{\text{pend}}}{g}},$$

preserves physical accuracy while maintaining algebraic closure.

7 Discussion

The *Complex π Equation* demonstrates that the geometric essence of π arises from algebraic phase symmetry rather than transcendence. By grounding circular closure in finite, measurable quantities and embedding their phase relation in complex space, the model unifies discrete and continuous curvature within one analytic framework. The invariance of S^* across polygonal doubling provides a precise and stable constant suitable for substitution in all equations traditionally dependent on π. Importantly, the surrogate achieves this without resorting to truncated decimal expansions: the doubling–invariance condition produces virtually the same value at each refinement (equal to within machine precision), so the entire sequence remains stable and free of transcendental approximation. The imaginary component of $z(\lambda)$ encodes the latent curvature phase—a mathematical representation of dimensional transition analogous to the brachistochrone descent. In this view, the circle is not an infinite limit but a closed resonance in complex space.

Beyond reproducing the circle, the finite-phase approach hints at broader applications. One could imagine analogous complex-phase embeddings for other transcendental constants or geometric quantities, where finite structures and phase constraints might replace infinite series. While speculative, such extensions underscore the potential of phase symmetry to unify discrete and continuous mathematics and invite further exploration.

8 Conclusion

The Complex π Equation establishes a closed, algebraic alternative to the transcendental definition of circular continuity. By constructing π from finite polygonal geometry and a single phase parameter, the formulation bridges discrete structure and continuous curvature without requiring an infinite process. The complex embedding $z(\lambda) = m + r[(1-2\lambda) + i\,2\sqrt{\lambda(1-\lambda)}]$ encapsulates both the real circumference surrogate and its imaginary phase complement, together forming a self-contained closure condition $(\Re z - m)^2 + (\Im z)^2 = r^2$. This reveals that what has historically been treated as an irrational constant is instead a manifestation of stable phase symmetry in the complex domain.

The result restores determinism to circular geometry: every refinement, phase adjustment, or complex rotation preserves the same invariant $|z|^2 = m_{\text{poly}}^2 + r^2$. The consistency of S^* across polygonal scales confirms that circular closure is inherently finite, not asymptotic. This opens a new analytical pathway where curvature, resonance, and continuity emerge from quantised geometry rather than transcendental approximation.

Unlike Richardson extrapolation or other convergence-acceleration schemes that combine successive approximations, the phase-invariance mechanism removes the dominant error term through a complex-phase constraint. This mechanism can be applied at finite polygonal resolution, reinforcing the view that curvature is an emergent property of algebraic phase symmetry rather than an artefact of infinite limits.

A Symbol Glossary

Symbol	Meaning	Units
A_q	Perimeter of the inscribed q-gon	dimensionless
B_q	Perimeter of the circumscribed q-gon	dimensionless
D	Normalised diameter of the circle (set to 1 in this work)	dimensionless
g	Gravitational acceleration	$\mathrm{m\,s^{-2}}$
L_{pend}	Pendulum length parameter in the period formula	m
m_{poly}	Midpoint $(A_q + B_q)/2$ of inscribed/circumscribed perimeters	dimensionless
q	Polygon refinement index or number of sides	dimensionless
r	Half-width $(B_q - A_q)/2$ between perimeters	dimensionless
$S(\lambda)$	Surrogate circumference ratio $= m_{\mathrm{poly}} + r(1 - 2\lambda)$	dimensionless
S^*	Stable surrogate obtained at λ^*	dimensionless
w	Quadratic image $w = z^2$ used to demonstrate invariants	dimensionless
$z(\lambda)$	Complex embedding $= m_{\mathrm{poly}} + r[(1 - 2\lambda) + i\,2\sqrt{\lambda(1-\lambda)}]$	dimensionless
λ	Phase parameter interpolating between inscribed and circumscribed states	dimensionless
λ^*	Phase parameter fixed by doubling invariance	dimensionless

B Acronyms

Acronym	Meaning
CPE	Complex π Equation

Acknowledgments

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